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# Dynamics-Based Attitude Determination Using the Global Positioning System

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**We present an algorithm for attitude determination using Global Positioning System (GPS) phase difierence measurements. The problem is formulated as an optimization problem on the group of rotation matrices subject to the system's kinematic and dynamic equations of motion. We first formulate an objective function whose update at each time step requires the integration of the dynamic equations and employs a forgetting factor to weight the history of accumulated GPS measurements. In the event that angular velocity measurements are not available, a cubic spline interpolation algorithm on the rotation group is used to estimate the angular velocity from the available attitude measurements. A geometric version of Newton's algorithm on the rotation group is also employed for the optimization, together with a method for rapidly calculating a suboptimal solution to be used as the initial point in the iteration. Simulation results comparing the performance of the proposed algorithm with those that neglect the dynamics are given.**

## Introduction

USE of the Global Positioning System (GPS) has now become widespread in applications ranging from land and air traffic control to surveying and aircraft precision landing. In this paper we address the problem of attitude determination using GPS signals and multiple antennas, with application to satellite and spacecraft attitude control. The proposed algorithms assume the availability of a model of the kinematic and dynamic equations for the rotating object whose attitude is to be determined.

Assuming cycle ambiguities have been resolved, the GPS-based attitude determination problem in its most fundamental form can be expressed as the minimization of a function on the three-dimensional rotation group SO(3) of the form

$$J(R) = \mathrm{tr}(RNR^TQ - 2RW) \tag{1}$$

where $R \in \mathrm{SO}(3)$ and $N$, $Q$, $W$ are given $3 \times 3$ matrices. The equation will be derived later from the measurement model. A related but simpler objective function that also arises in the context of attitude estimation is that of the classic Wahba problem,[1] which can be posed as the minimization of the function

$$J(R) = \mathrm{tr}(RW) \tag{2}$$

where $W$ is a given nonsingular $3 \times 3$ matrix constructed from sensor measurements. Although closed-form solutions involving the singular value decomposition (SVD)[2] of $W$ can be constructed for Wahba's objective function, it does not appear that similar closed-form solutions can be obtained for the more general objective function.

For the general objective function of Eq. (1), Cohen[2] developed a recursive algorithm for on-line estimation of the attitude that is based on a linearization of the objective function. Axelrad and Behre[3] discussed and compared several time and frequency domain algorithms for estimating spinning spacecraft angular rates and orientation of the angular momentum vector using GPS phase data. More recently Bar-Itzhack et al.[4] have proposed iterative quaternion-based algorithms for GPS-based attitude determination, which have been generalized by Crassidis and Markley[5] for the case of multiple baseline vectors. Park et al.[6] also proposed a set of geometric numerical optimization algorithms for the objective function of Eq. (1).

In addition to the preceding methods that are based on a deterministic parameter optimization approach, model-based approaches to attitude determination have also been proposed in the literature. Lefferts et al.[7] presented a dynamic model-based Kalman filter for spacecraft attitude estimation, whereas Axelrad and Ward[8] developed an extended Kalman filter for GPS-based attitude determination using the quaternion representation for rotations. Crassidis and Markley[9,10] also presented a class of quaternion-based attitude estimation algorithms and predictive filters, which involve the kinematics and dynamics, and proposed a predictive estimation algorithm based on propagating the kinematic equations.[11]

In this paper we present an algorithm for attitude determination that also minimizes the objective function of Eq. (1), but makes use of both the available kinematic and dynamic models for the rotating object. The algorithm possesses several novel features. First, the objective function is reformulated such that at each discrete time step it is updated with a forgetting factor to weight appropriately the history of accumulated GPS measurements. (The Kalman filter also weights measurements, but is restricted to systems in which the state space is linear; for our problem the state space SO(3) is nonlinear, and as such does not admit a simple, geometrically well-defined notion of Gaussian noise—see in this regard Liao[12] and McConnell.[13]) The update requires the integration of the dynamic equations of motion in an essential way using the most current values of the attitude and angular velocity. In the event that angular velocity measurements are not available, we adopt a recently developed cubic-spline interpolation algorithm on SO(3) to estimate the angular velocity based on the attitude data.[14,15]

The numerical optimization at each discrete time step can be performed using any number of optimization algorithms. Here we use the geometric version of Newton's algorithm on SO(3) first developed in Park et al.[6] Independent of the choice of algorithms, however, numerical optimization algorithms in general are quite sensitive to the choice of initial condition. In this paper we develop a method for the rapid calculation of a suboptimal solution that provides an effective choice for the initial starting point. We first obtain a minimum for the objective function in the enlarged space of real $3 \times 3$ matrices, then project this solution to SO(3). The resulting algorithm makes systematic use of both the history of GPS measurements and kinematic and dynamic models, and simulation results show significantly improved performance over conventional parameter optimization approaches that neglect the dynamics.

The paper is organized as follows. We first formulate the attitude-determination problem under the assumption that a model of the kinematics and dynamics of the rotating object (modeled as a rigid body) are available. We then present a method for rapidly obtaining a suboptimal solution to the problem, followed by a description of

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the overall algorithm. Then simulation results demonstrating the performance of the algorithm are given, together with a summary and suggestions for further performance improvement.

## Problem Definition

For the purposes of this paper, we express the dynamics of a rigid body in three-dimensional space as

$$M\dot{\omega} = [\omega]M\omega + u + \xi, \qquad \dot{R} = R[\omega] \tag{3}$$

where $R \in \mathrm{SO}(3)$, $M$ is the inertia tensor, $u$ is the control torque, $\xi$ is the white Gaussian input noise, $\omega$ is the angular velocity in the body-fixed reference frame, and $[\omega]$ is its $3\times 3$ skew-symmetric matrix representation, that is,

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

It can be easily verified that $[\omega]v = \omega \times v$ for $v \in \mathbb{R}^3$. Crouch[16] contains extensive descriptions of the form of the dynamic equations for satellites driven by gas jets and momentum wheels.

We make the following assumptions about the GPS measurements. First, the direction vectors to each of the GPS satellites are assumed to be known accurately. Second, the baselines formed from the GPS receiver antennas are known with sufficient accuracy. Third, integer ambiguities are assumed to be resolved in some appropriate fashion. Under these assumptions Cohen[2] suggests the following GPS measurement model:

$$z_{ij} = \hat{s}_j^T R b_i + v_{ij}$$

where $\hat{s}_j$ is the unit vector from the origin of the master antenna to the $j$th satellite (represented with respect to the local reference frame), $b_i$ is the $i$th baseline vector (described in the body frame), $R$ is the rotation matrix describing the orientation of the local reference frame relative to the body frame, and $z_{ij}$ is the actual measurement of the inner product of $\hat{s}_j$ and $b_i$ (Fig. 1). Because of the uncertainty in the phase measurements, $z_{ij}$ is expressed as the sum of the ideal value of $\hat{s}_j^T R b_i$ and additive zero-mean white Gaussian noise $v_{ij}$.

For convenience, we arrange the measurement data and noise in matrix form:

$$Z = \begin{pmatrix} z_{11} & \cdots & z_{1m} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nm} \end{pmatrix}, \qquad V = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix}$$

The GPS measurement model can then be written as

$$Z = S^T R B + V \tag{4}$$

where $B = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^{3\times m}$ and $S = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n) \in \mathbb{R}^{3\times n}$.

We now cast the attitude-determination problem as an optimization problem on the space of rotation matrices SO(3) by formulating an objective function that takes into account the dynamics (3) as well as the measurement model (4). Our algorithm is formulated under the following assumptions about the operation of the satellite. First, the attitude trajectory of the satellite is assumed to be reasonably smooth, as is the trajectory of the angular velocity of the satellite (i.e., one can assume the sampling interval for the GPS measurements is below Nyquist's upper limit). The inertia tensor of the satellite $M$, the baseline vectors $b_i$, and the direction vectors to the GPS satellites $\hat{s}_j$ are assumed to be known with sufficient accuracy. Under these assumptions we consider separately the case when the angular velocity $\omega$ can be obtained from a rate sensor and the GPS receiver-only case, when $\omega$ is unknown and must be estimated from the earlier attitude measurements.

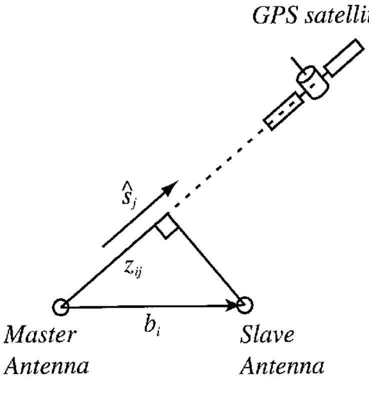


**Fig. 1 Meanings of $z_{ij}$, $\hat{s}_j^T$, and $b_i$.**

## Objective Function Formulation

According to Park et al.,[6] the objective function

$$\min_{R\in \mathrm{SO}(3)} \sum_{i=1}^{m}\sum_{j=1}^{n}\left(z_{ij} - \hat{s}_j^T R b_i\right)^2$$

proposed by Cohen[2] can be arranged into the following form:

$$\begin{aligned} J(R) &= \mathrm{tr}(Z - S^T R B)^T (Z - S^T R B) \\ &= \mathrm{tr}(QR^T NR - 2WR) + \text{const} \end{aligned} \tag{5}$$

where $Q = BB^T$ and $N = SS^T$ are $3\times 3$ symmetric matrices and $W = BZ^T S^T$ is a general $3\times 3$ matrix, and $B$ and $S$ are as already defined. In Cohen's objective function the physical meaning of $z_{ij}$, $\hat{s}_j^T$, and $b_i$ are the same as described in the preceding section.

Let $J_0$ and $J_1$ denote values of the preceding objective function taken at two consecutive time intervals $t_0$ and $t_1$:

$$J_0 = \mathrm{tr}\left(R_0 Q R_0^T N_0 - 2W_0 R_0\right) \tag{6}$$

$$J_1 = \mathrm{tr}\left(R_1 Q R_1^T N_1 - 2W_1 R_1\right) \tag{7}$$

$J_0$ and $J_1$ can in turn be regarded as objective functions whose terms consist of measurement data taken at consecutive times $t_0$ and $t_1$, respectively. Observe that $Q = BB^T$ is constant over time, whereas $N_i = S_i S_i^T$ and $W_i = BZ_i^T S_i^T$ vary with time.

The actual attitude at time $t_0$, denoted $R_0$, and the actual attitude at time $t_1$, denoted $R_1$, are related according to the dynamics equation (3). Let $R_1 = R_{u,1}R_0$ for some rotation matrix $R_{u,1}$. Then $R_0 = R_{u,1}^T R_1$, and the objective function (6) at time $t_0$ is given by

$$J_0 = \mathrm{tr}\left(R_{u,1}^T R_1 Q R_1^T R_{u,1} N_0 - 2W_0 R_{u,1}^T R_1\right)$$

The objective function at time $t_0$ is now described as a function of $R_1$, the attitude of the satellite at time $t_1$. Now, because $J_0$ is represented in terms of $R_1$ we combine the two objective functions (6) and (7) into an accumulated objective function defined at $t_1$ as follows:

$$\begin{aligned} J_1^a &= \mu J_0 + J_1 \\ &= \mathrm{tr}\left(R_1 Q R_1^T N_1^a - 2W_1^a R_1\right) \end{aligned} \tag{8}$$

where $N_1^a = \mu R_{u,1} N_0 R_{u,1}^T + N_1$, $W_1^a = \mu W_0 R_{u,1}^T + W_1$, and $\mu$ is a constant parameter such that $0 < \mu \le 1$ (we describe its physical meaning shortly).

To accumulate the GPS measurement data at time $t_2$, it is sufficient to combine the accumulated objective function at time $t_1$, Eq. (8), and a new objective function, which consists of the GPS measurement data at time $t_2$ only, because Eq. (8) already reflects the measurement data at time $t_0$ and $t_1$:

$$\begin{aligned} J_2^a &= \mu J_1^a + J_2 \\ &= \mathrm{tr}\left(R_2 Q R_2^T N_2^a - 2W_2^a R_2\right) \end{aligned}$$

where $N_2^a = \mu R_{u,2} N_1^a R_{u,2}^T + N_2$ and $W_2^a = \mu W_1^a R_{u,2}^T + W_2$.

Repeating this procedure, the accumulated objective function at time $t_i$ is given as follows:

$$\begin{aligned} J_i^a &= \mu J_{i-1}^a + J_i \\ &= \mathrm{tr}\left(R_i Q R_i^T N_i^a - 2W_i^a R_i\right) \end{aligned} \tag{9}$$

where $N_i^a = \mu R_{u,i} N_{i-1}^a R_{u,i}^T + N_i$ and $W_i^a = \mu W_{i-1}^a R_{u,i}^T + W_i$.

The parameter $\mu$ is called the *forgetting factor* or *discounting factor* (see, e.g., Åström and Wittenmark[17]). The most recent data are given unit weight, but data that are $n$ time units old are weighted by $\mu^n$. The method is therefore referred to as *exponential forgetting* or *exponential discounting.* The forgetting factor is given by

$$\mu = e^{-h/T_f}$$

$$
\begin{aligned}
J_0^a &= J_0 \\
&= \mathrm{tr}(R_0 Q R_0^{\mathrm{T}} N_0^a - 2W_0^a R_0)
\end{aligned}
\quad \longleftarrow \quad
\begin{cases}
N_0^a = N_0 \\
W_0^a = W_0
\end{cases}
$$

$$\downarrow$$

$$
\begin{aligned}
J_1^a &= \mu J_0^a + J_1 \\
&= \mathrm{tr}(R_1 Q R_1^{\mathrm{T}} N_1^a - 2W_1^a R_1)
\end{aligned}
\quad \longleftarrow \quad
\begin{cases}
N_1^a = \mu R_{u,1} N_0^a R_{u,1}^{\mathrm{T}} + N_1 \\
W_1^a = \mu W_0^a R_{u,1}^{\mathrm{T}} + W_1
\end{cases}
$$

$$\downarrow$$

$$\vdots$$

$$\downarrow$$

$$
\begin{aligned}
J_i^a &= \mu J_{i-1}^a + J_i \\
&= \mathrm{tr}(R_i Q R_i^{\mathrm{T}} N_i^a - 2W_i^a R_i)
\end{aligned}
\quad \longleftarrow \quad
\begin{cases}
N_i^a = \mu R_{u,i} N_{i-1}^a R_{u,i}^{\mathrm{T}} + N_i \\
W_i^a = \mu W_{i-1}^a R_{u,i}^{\mathrm{T}} + W_i
\end{cases}
$$

$$\downarrow$$

$$\vdots$$

**Fig. 2 Flowchart of the accumulation procedure of the GPS measurement data.**

where $h$ is the sampling period and $T_f$ is the time constant for the exponential forgetting. The time constant $T_f$ is typically determined from off-line simulations. $R_{u,i} = R_i \hat{R}_{i-1}^T$, and $R_i$ is obtained by numerically integrating the dynamics equation (3) from $t_{i-1}$ to $t_i$ with the initial conditions $\hat{R}_{i-1}$ and $\hat{\omega}_{i-1}$. $\hat{R}_{i-1}$ is calculated at time $t_{i-1}$ by minimizing $J_{i-1}^a$, whereas $\hat{\omega}_{i-1}$ is assumed to be obtained using a suitable rate sensor if one is available. In the event that measurement of $\hat{\omega}_{i-1}$ is not possible, we can estimate it using any number of extrapolation methods; in this paper we consider one based on the SO(3) cubic-spline interpolation algorithm presented in Refs. 14 and 15.

The GPS measurement data accumulation procedure can be described by the following recursive equations:

$$
\begin{aligned}
N_i^a &= \mu R_{u,i} N_{i-1}^a R_{u,i}^T + N_i \\
W_i^a &= \mu W_{i-1}^a R_{u,i}^T + W_i, \qquad (i = 1, 2, \ldots)
\end{aligned}
$$

with initial values $N_0^a = N_0$ and $W_0^a = W_0$. $N_i^a$ is symmetric because $N_{i-1}^a$, and $N_i$ and $N_0$ are symmetric. Hence, the objective function (9) is of the same form as Eq. (5).

The accumulation procedure of the GPS measurement data is summarized in Fig. 2.

## Determining an Initial Guess

As is characteristic of most iterative optimization methods, the starting point is critical in determining the convergence properties of the optimization procedure. We now develop a method for determining a suitable starting point in the optimization procedure. The starting point is calculated from only a few matrix multiplications and one SVD of a $3 \times 3$ matrix and improves significantly the convergence behavior of the optimization algorithm.

The starting point is obtained by enlarging the search space from SO(3) to the space of all real $3 \times 3$ matrices. In this case the solution can be readily found from the first-order necessary conditions. The following result is useful in this regard.

*Proposition:* Let $N$ and $Q$ be symmetric positive definite $3 \times 3$ matrices and $W$ an arbitrary $3 \times 3$ real matrix. The real $3 \times 3$ matrix $R$ that minimizes

$$J(R) = \mathrm{tr}(RQR^T N - 2WR)$$

is given by $\bar{R} = N^{-1} W^T Q^{-1}$.

*Proof:* The first-order necessary condition is given by $QR^T N - W = 0$, from which it immediately follows that $\bar{R} = N^{-1} W^T Q^{-1}$ is the only possible extremum. To show that it is a minimum, observe that the objective function is quadratic in $R$. Therefore the Hessian of $J$ simply corresponds to the quadratic term of $J$, and it suffices to show that $\mathrm{tr}(RQR^T N) > 0$ for all arbitrary $R$. Because $N$ is positive definite, there exists some $M \in \mathbb{R}^{3 \times 3}$ such that $N = MM^T$. Using the relation $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$, we can write $\mathrm{tr}(RQR^T N) = \mathrm{tr}(RQR^T MM^T) = \mathrm{tr}(M^T RQR^T M)$. From the positive definiteness of $Q$, it follows that this must be greater than zero for arbitrary $R$, thereby establishing that $\bar{R} = N^{-1} W^T Q^{-1}$ is a global minimum for $J(R)$. □

We now examine the requirement that both $N$ and $Q$ be positive definite. Because $N = SS^T$ and $Q = BB^T$, the preceding requirement can be satisfied if and only if $\mathrm{rank}(S) = \mathrm{rank}(B) = 3$. Physically the first requirement implies that the GPS satellites must be chosen in such a way that the direction vectors are not coplanar. The second requirement on the rank of $B$ can be easily satisfied by noting that it is formed from the baseline vectors, which are determined by the designer of the satellite. By proper design of the satellite and an appropriate choice of GPS satellites, the nonsingularity of $N$ and $Q$, and hence the existence of $\bar{R}$, can be ensured.

Another useful physical interpretation of $\bar{R}$ involving probability is that it can be regarded as an unbiased estimator to the actual attitude $R$ in the following sense. Because $W = BZ^T S^T$ and $Z = S^T RB + V$,

$$
\begin{aligned}
\bar{R} &= N^{-1} SS^T RBB^T Q^{-1} + N^{-1} SVB^T Q^{-1} \\
&= R + N^{-1} SVB^T Q^{-1}
\end{aligned}
$$

Taking expectations on both sides,

$$\mathcal{E}[\bar{R}] = \mathcal{E}[R] + N^{-1} S \underbrace{\mathcal{E}[V]}_{=0} B^T Q^{-1} = \mathcal{E}[R]$$

because the elements of $V$ are assumed to be zero mean.

$\bar{R} = N^{-1} W^T Q^{-1}$ is *not* an element of SO(3) in general, that is, $\bar{R}^T \bar{R} \neq I$ and $\det \bar{R} \neq 1$. We therefore search for the attitude matrix $R_{\mathrm{init}} \in \mathrm{SO}(3)$ that is closest to $\bar{R}$ in the Euclidean sense. This leads to the classical Wahba problem on SO(3), one whose solution can be characterized completely:

$$\min_{R_{\mathrm{init}} \in \mathrm{SO}(3)} \mathrm{tr}(R_{\mathrm{init}} - \bar{R})^T (R_{\mathrm{init}} - \bar{R}) = \min_{R_{\mathrm{init}} \in \mathrm{SO}(3)} \left[6 - 2\mathrm{tr}\left(\bar{R}^T R_{\mathrm{init}}\right)\right] \tag{10}$$

The unique solution can be regarded as the projection of $\bar{R}$ onto SO(3) and is given by

$$R_{\mathrm{init}} = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |\bar{R}^T| \end{pmatrix} U^T \tag{11}$$

where $|\cdot|$ denotes the determinant, and $U$, $\Sigma$, and $V$ are obtained from the singular value decomposition of $\bar{R}^T$, i.e., $\bar{R}^T = U \Sigma V^T$.

## Recursive Algorithm

We now describe the general recursive algorithm for attitude estimation based on the preceding results. We assume that $\hat{R}_{i-1}$, the estimate of the attitude at time $t_{i-1}$, and $N_{i-1}^a$, $W_{i-1}^a$ are available. As will be evident in the following, the algorithm varies slightly depending on whether or not angular velocity measurements are available.

*Step 1:* With the initial conditions $R(t_{i-1}) = \hat{R}_{i-1}$ and $\boldsymbol{\omega}(t_{i-1}) =$ measured or interpolated value, integrate the following dynamic equations from $t_{i-1}$ to $t_i$ to obtain $R_{u,i}$ assuming known input torque $\boldsymbol{u}$:

$$M\dot{\boldsymbol{\omega}} = [\boldsymbol{\omega}]M\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{\xi}, \qquad \dot{R} = R[\boldsymbol{\omega}]$$

Then $R_{u,i} = R(t_i)\hat{R}_{i-1}^T$. Integrating the white Gaussian noise

$$\int_{t_{i-1}}^{t_i} \boldsymbol{\xi}(\tau)\,\mathrm{d}\tau$$

yields 0 because of the mean-ergodicity property.[18]

*Step 2:* Update $N_i^a$ and $W_i^a$ using the value of $R_{u,i}$ obtained from Step 1:

$$N_i^a = \mu R_{u,i} N_{i-1}^a R_{u,i}^T + N_i, \qquad W_i^a = \mu W_{i-1}^a R_{u,i}^T + W_i$$

*Step 3:* Determine $\hat{R}_i$, the optimal attitude estimate at $t_i$, by minimizing the following objective function:

$$J_i^a(\hat{R}_i) = \operatorname{tr}\left(\hat{R}_i Q \hat{R}_i^T N_i^a - 2W_i^a \hat{R}_i\right)$$

The starting point for the optimization procedure is obtained from the suboptimal projection procedure described in the preceding section. In the event that $Q$ or $N$ is nearly singular because of the selected baseline or sightline configuration, $\hat{R}_{i-1}$ is a reasonable candidate for the starting point.

*Step 4:* Go to Step 1 with the new attitude estimate obtained from Step 3.

## With Angular Velocity Measurements

When angular velocity measurements are available, we can obtain $R_{u,i}$ of Step 1 by simply integrating the kinematic equation $\dot{R} = R[\boldsymbol{\omega}]$ because gyro measurements are typically far more accurate than the dynamic model propagation.

## Without Angular Velocity Measurements

To integrate the dynamic equations of Step 1 from $t_{i-1}$ to $t_i$, the angular velocity at $t_{i-1}$ is required. If angular velocity measurements are not available, an estimation procedure based on the preceding measurements of attitude and the satellite dynamics must be developed. In principle any interpolation or extrapolation scheme on SO(3) will do; here we adopt a method based on the SO(3) cubic-spline interpolation algorithm developed by Park and Ravani[14] and Kang and Park.[15] For small time intervals it is argued by Kang and Park[15] that the resulting cubic-spline trajectory in SO(3) minimizes angular acceleration, in the same sense that cubic splines in Euclidean space minimize linear acceleration.

Given an ordered set of knot points $\{R_0, R_1, \ldots, R_N\}$ in SO(3) together with the corresponding knot times $\{t_0, t_1, \ldots, t_N\}$, the algorithm generates a twice differentiable curve $R(t)$ passing through the knot points in the correct order. The angular velocity $\boldsymbol{\omega}(t)$ can also be analytically computed from the resulting $R(t)$. For our problem we interpolate $\{R_{t_{i-n}}, R_{t_{i-n+1}}, \ldots, R_{t_i}\}$ to obtain $\boldsymbol{\omega}_{t_i}$. The interpolation algorithm proceeds according to the following steps:

*Given*

$$\{R_0, R_1, \ldots, R_N\} = \text{knot points}, \qquad \{t_0, t_1, \ldots, t_N\} = \text{knot times}$$

$$\boldsymbol{\omega}_0 = \text{angular velocity at } t_0 \text{ in moving frame coordinates}$$

$$\boldsymbol{\alpha}_0 = \text{angular acceleration at } t_0 \text{ in moving frame coordinates}$$

$$(\text{Condition:} \quad \operatorname{tr}\left(R_{i-1}^T R_i\right) \neq -1 \qquad \text{for} \qquad i = 1, 2, \ldots, N)$$

*Preprocessing* for $i = 1$ to $N$ do

$$[\boldsymbol{r}_i] = \log\left(R_{i-1}^T R_i\right)$$

$$A_i = I - \frac{1 - \cos\|\boldsymbol{r}_i\|}{\|\boldsymbol{r}_i\|^2}[\boldsymbol{r}_i] + \frac{\|\boldsymbol{r}_i\| - \sin\|\boldsymbol{r}_i\|}{\|\boldsymbol{r}_i\|^3}[\boldsymbol{r}_i]^2$$

*Initialization*

$$\boldsymbol{c}_1 = \boldsymbol{\omega}_0, \qquad \boldsymbol{b}_1 = \boldsymbol{\alpha}_0/2, \qquad \boldsymbol{a}_1 = \boldsymbol{r}_1 - \boldsymbol{b}_1 - \boldsymbol{c}_1$$

*Iteration* for $i = 2$ to $N$ do

$\boldsymbol{s} = \boldsymbol{r}_i$ (temporary variable)

$\boldsymbol{t} = 3\boldsymbol{a}_{i-1} + 2\boldsymbol{b}_{i-1} + \boldsymbol{c}_{i-1}$ (temporary variable)

$\boldsymbol{u} = 6\boldsymbol{a}_{i-1} + 2\boldsymbol{b}_{i-1}$ (temporary variable), $\qquad \boldsymbol{c}_i = A_{i-1}\boldsymbol{c}_{i-1}$

$$\boldsymbol{b}_i = \frac{1}{2}\left\{\boldsymbol{u} - \frac{\boldsymbol{s}^T\boldsymbol{t}}{\|\boldsymbol{s}\|^4}(2\cos\|\boldsymbol{s}\| + \|\boldsymbol{s}\|\sin\|\boldsymbol{s}\| - 2)(\boldsymbol{s}\times\boldsymbol{t}) - \frac{1-\cos\|\boldsymbol{s}\|}{\|\boldsymbol{s}\|^2}(\boldsymbol{s}\times\boldsymbol{u}) + \frac{\boldsymbol{s}^T\boldsymbol{t}}{\|\boldsymbol{s}\|^5}(3\sin\|\boldsymbol{s}\| - \|\boldsymbol{s}\|\cos\|\boldsymbol{s}\| - 2\|\boldsymbol{s}\|) \times [\boldsymbol{s}\times(\boldsymbol{s}\times\boldsymbol{t})] + \frac{\|\boldsymbol{s}\| - \sin\|\boldsymbol{s}\|}{\|\boldsymbol{s}\|^3}[\boldsymbol{t}\times(\boldsymbol{s}\times\boldsymbol{t}) + \boldsymbol{s}\times(\boldsymbol{s}\times\boldsymbol{u})]\right\}$$

$\boldsymbol{a}_i = \boldsymbol{s} - \boldsymbol{b}_i - \boldsymbol{c}_i$

*Results* for $t_{i-1} \leq t \leq t_i$

$$\tau = \frac{t - t_{i-1}}{t_i - t_{i-1}}, \qquad R(t) = R_{i-1}\exp\left[a_i\tau^3 + b_i\tau^2 + c_i\tau\right]$$

and the angular velocity at time $t_N$ is

$$\hat{\boldsymbol{\omega}}_N = A_N(3\boldsymbol{a}_N + 2\boldsymbol{b}_N + \boldsymbol{c}_N)$$

The matrix logarithm on SO(3) invoked in the preprocessing stage can be computed analytically from the formula

$$\log R = (1/2\sin\phi)(R - R^T)$$

where $\phi$ satisfies $1 + 2\cos\phi = \operatorname{tr}(R)$. See Kang and Park[15] for the details of the derivation and numerical performance results.

## Simulation Results

We now compare the performance of the dynamics-based attitude-determination algorithm with the static measurement-based algorithm developed by Park et al.[6] For both algorithms we use an identical set of GPS measurement data. Simulations are performed under two distinctive cases: in the first case the attitude-determination algorithm is simulated with a satellite, which has a rate sensor and a GPS receiver. In the second case the satellite is assumed to be equipped with a GPS receiver only, and the angular velocity of the satellite is estimated from the preceding attitude estimates using the cubic-spline interpolation algorithm on SO(3).

The satellite is modeled as a rectangular body of dimension $0.5 \times 0.5 \times 1$ m, with a mass of 800 kg and rotating about a fixed axis at 5 rpm. Its moments and products of inertia are calculated to be $I_{xx} = I_{yy} = 83.33\ \mathrm{kg \cdot m^2}$, $I_{zz} = 33.33\ \mathrm{kg \cdot m^2}$. We assume that the baseline matrix $B$ is the $3 \times 3$ identity matrix, and the sightline matrix $S$ is generated pseudorandomly to prevent $S$ from being (nearly) singular.

**Table 1 Summary of the simulation results for the rate sensor-aided case. For each result the three lines represent the roll, pitch, and yaw errors, respectively.**

| Time step, s | Noise variance | Dynamics-based determination | Static measurement-only minimization |
|---|---|---|---|
| 0.1 | 0.01 | 0.897 | 3.826 |
| | | 0.701 | 3.413 |
| | | 0.914 | 3.810 |
| 0.1 | 0.001 | 0.163 | 1.331 |
| | | 0.187 | 1.059 |
| | | 0.150 | 1.124 |
| 0.5 | 0.01 | 0.503 | 3.690 |
| | | 0.554 | 3.033 |
| | | 0.927 | 3.582 |
| 0.5 | 0.001 | 0.226 | 1.641 |
| | | 0.087 | 1.004 |
| | | 0.152 | 1.457 |
| 1 | 0.01 | 0.742 | 3.754 |
| | | 1.082 | 3.846 |
| | | 0.636 | 4.347 |
| 1 | 0.001 | 0.182 | 1.276 |
| | | 0.308 | 1.435 |
| | | 0.213 | 1.412 |
| 5 | 0.01 | 0.519 | 2.969 |
| | | 0.859 | 4.877 |
| | | 0.953 | 4.638 |
| 5 | 0.001 | 0.200 | 0.943 |
| | | 0.252 | 1.043 |
| | | 0.215 | 1.111 |

The block diagram of the simulation and the dynamics-based attitude-determination algorithm is shown in Fig. 3.

**Case 1: With Angular Velocity Measurements**

We assume the angular velocity of the satellite can be measured with sufficient accuracy with the aid of rate sensors. For this case the forgetting factor $\mu$ is set to 1, that is, no forgetting is applied.

Figure 4 shows the roll-pitch-yaw error in degrees when the GPS measurement data are assumed to be collected every 0.5 s and the measurement noise variance is 0.001. The solid line is the error of the dynamics-based determination algorithm, and the dotted line is that of the static measurement-only minimization. The static measurement-only minimization is simulated by using the preceding step's attitude estimate $\hat{R}_i$ as the starting point of the iteration at time $t_{i+1}$.

Table 1 summarizes the rms of the estimation errors under various simulation conditions. In each case in the table, the three lines represent the roll, pitch, and yaw errors, respectively, in degrees. As the results show, the accuracy of the estimation depends more on the noise variance rather than the measurement time step when the

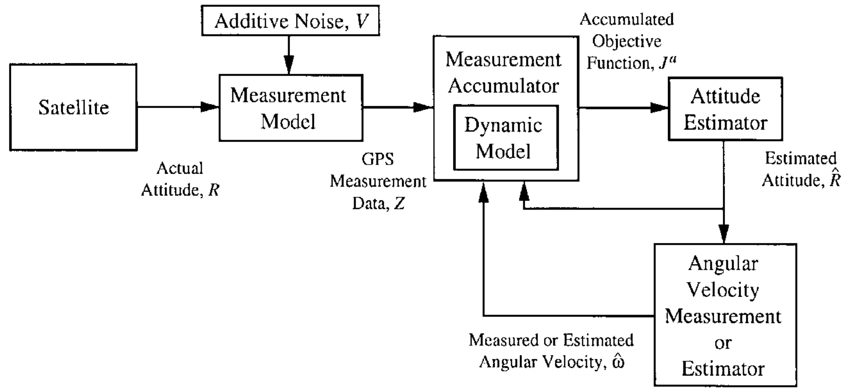


**Fig. 3 Block diagram of the simulation.**

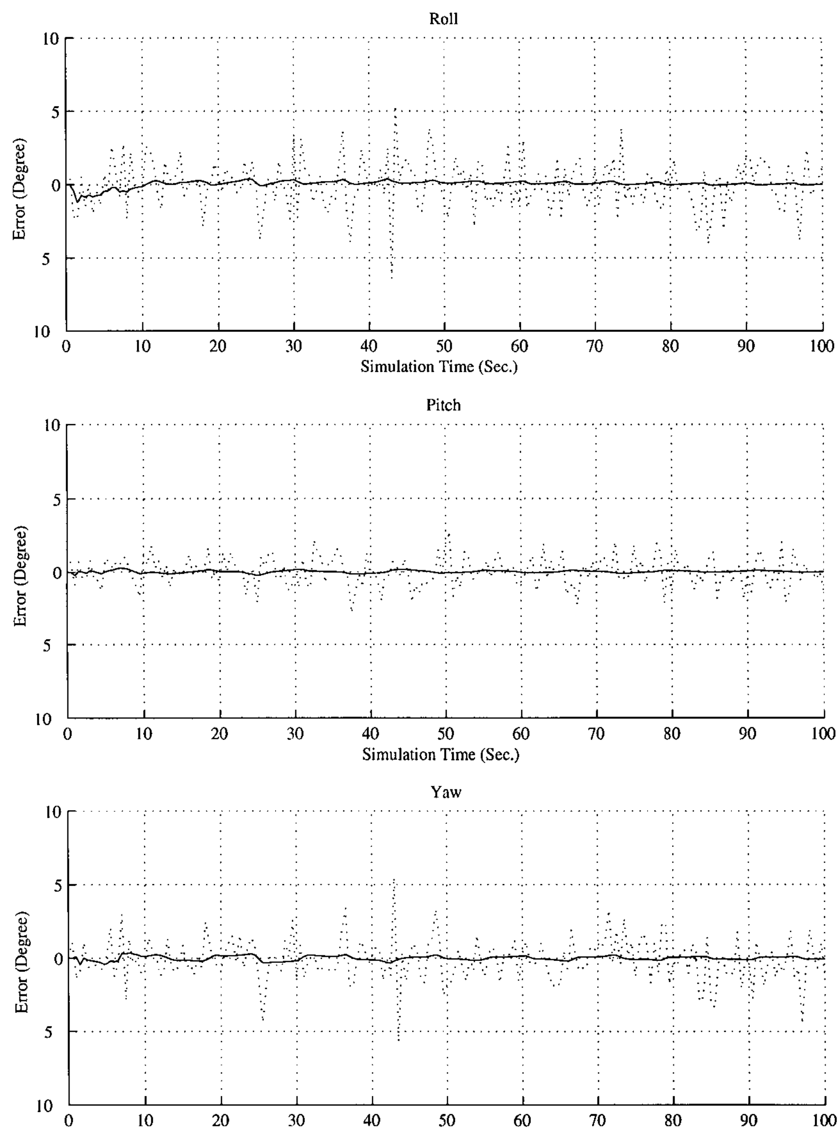


**Fig. 4 Attitude estimation error when the time step is 0.5 s and measurement noise variance is 0.001: ——, dynamics-based estimation; ·····, static measurement-only estimation.**

angular velocity is measured accurately. However, as the time step is increased, the choice of the starting point becomes more and more critical because the objective function on SO(3) will in general have more than one local minima.

**Case 2: Without Angular Velocity Measurements**

In the event that angular velocity measurements are not available, we use the accumulated GPS measurement data already obtained to estimate the angular velocity based on the cubic-spline interpolation algorithm on SO(3). To examine the required accuracy of the dynamic model, we simulate the dynamics-based algorithm with a model that contains estimation errors in the inertia matrix $M$ and compare it to the results obtained from the static measurement-only minimization algorithm. The simulation results of Table 2 suggest that a conservative estimate of $M$ to within 15% of their actual values results in satisfactory performance of the dynamics-based algorithm. In Table 2 the error of the inertia matrix $M$ is defined as $\|M_{\text{actual}} - M_{\text{model}}\|/\|M_{\text{actual}}\|$, where $\|\cdot\|$ denotes the matrix Frobenius norm $\|A\| = \sqrt{(\sum_{i,j} a_{ij}^2)}$, and the attitude error is defined by the natural distance metric on SO(3): $d(R_{\text{actual}}, R_{\text{estimate}}) \triangleq \|\log(R_{\text{actual}}^T R_{\text{extimate}})\|$ (see Park[19]). The error figures were obtained as the average over 20 trials for each case converted to degrees.

In Fig. 5 the solid lines show the attitude estimation error of the dynamics-based algorithm, whereas the dotted lines show that of the static measurement-only minimization. For this simulation the time step is set to 0.1 s, the noise variance to 0.01, and there is no forgetting, that is, $\mu$ is set to 1. As evident from the graph, the rms of the estimation error of the dynamics-based algorithm is smaller than that of the static measurement-only minimization by nearly an order. Figure 6 shows the component-wise angular velocity estimation errors corresponding to the results shown in Fig. 5.

To demonstrate the importance of the starting point in the optimization procedure, the static measurement-only minimization is simulated with two different choices of starting point. The results in the column labeled "static measurement-only minimization (1)" in Table 3 are obtained by letting the starting point be the optimal

**Table 2 Average attitude error of the dynamics-based estimation algorithm for uncertainty in the values of the inertia matrix *M***

| Percentage in the error matrix inertia terms, % | Average attitude error for dynamics-based estimation, deg | Average attitude error for static measurement-only estimation, deg | Ratio column 2/ column 3 |
|---|---|---|---|
| 1 | 1.039 | 5.554 | 0.187 |
| 3 | 1.263 | 5.554 | 0.227 |
| 5 | 1.718 | 5.554 | 0.309 |
| 7 | 2.113 | 5.554 | 0.380 |
| 9 | 2.544 | 5.554 | 0.458 |
| 11 | 3.379 | 5.554 | 0.608 |
| 13 | 4.254 | 5.554 | 0.766 |
| 15 | 4.876 | 5.554 | 0.878 |

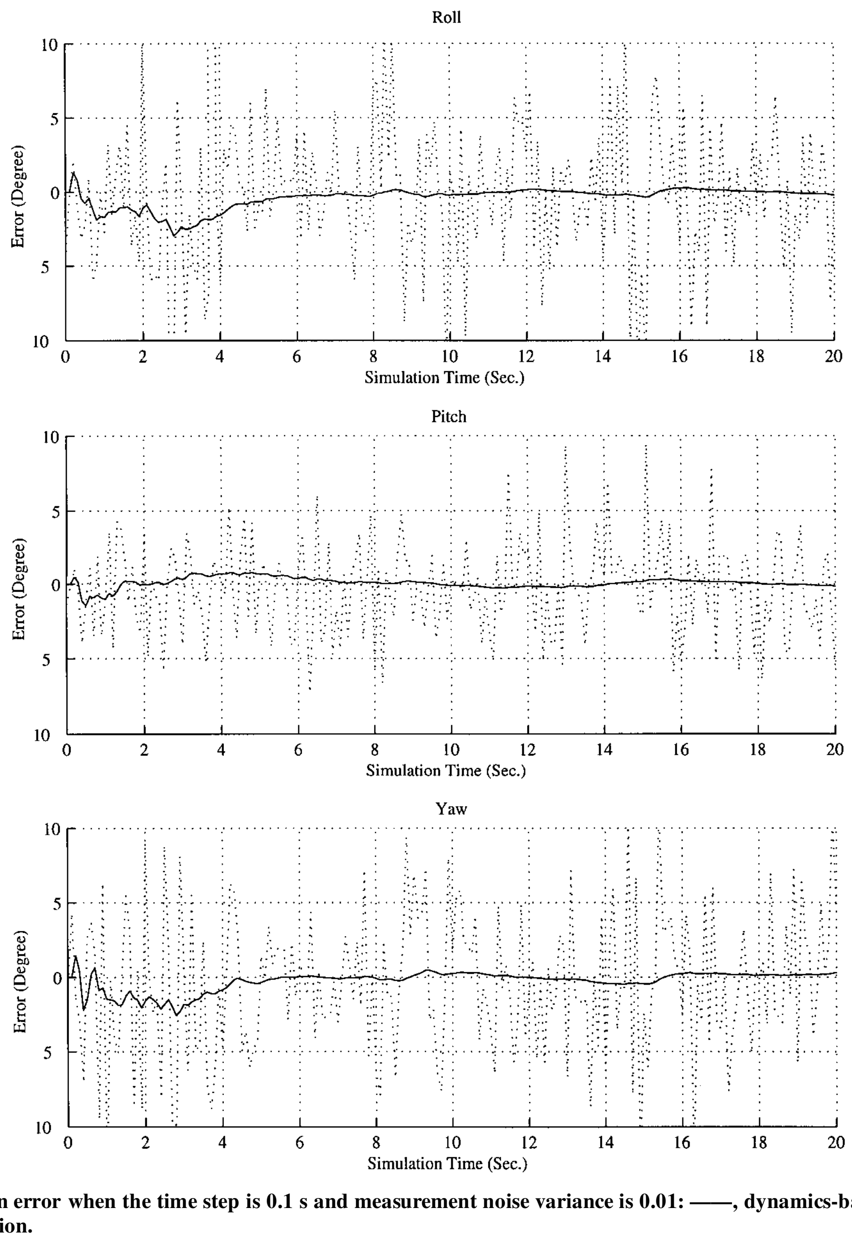


**Fig. 5 Attitude estimation error when the time step is 0.1 s and measurement noise variance is 0.01: ——, dynamics-based estimation; ····, static measurement-only estimation.**

**Table 3 Summary of the simulation results for the GPS receiver-only case. For each result the three lines represent the roll, pitch, and yaw errors, respectively**

| Time step, s | Noise variance | Forgetting factor | Dynamics-based determination | Static measurement-only minimization (1) | Static measurement-only minimization (2) |
|---|---|---|---|---|---|
| 0.1 | 0.01 | 1 | 0.423 | 3.090 | 5.820 |
| | | | 0.501 | 3.505 | 5.327 |
| | | | 0.470 | 3.309 | 5.019 |
| 0.1 | 0.001 | 1 | 0.140 | 1.005 | 1.963 |
| | | | 0.197 | 1.216 | 1.907 |
| | | | 0.193 | 1.248 | 2.337 |
| 0.3 | 0.01 | 0.5 | 2.907 | 2.924 | 5.051 |
| | | | 3.453 | 3.354 | 5.235 |
| | | | 3.497 | 3.481 | 5.133 |
| 0.3 | 0.001 | 0.5 | 0.024 | 0.998 | 1.120 |
| | | | 1.243 | 1.339 | 1.438 |
| | | | 1.200 | 1.245 | 1.300 |
| 0.5 | 0.01 | 0.2 | 3.451 | 3.420 | 5.666 |
| | | | 3.282 | 3.340 | 5.462 |
| | | | 5.768 | 5.952 | 5.187 |
| 0.5 | 0.001 | 0.2 | 1.402 | 1.298 | 2.915 |
| | | | 0.985 | 0.973 | 2.249 |
| | | | 1.290 | 1.283 | 2.462 |

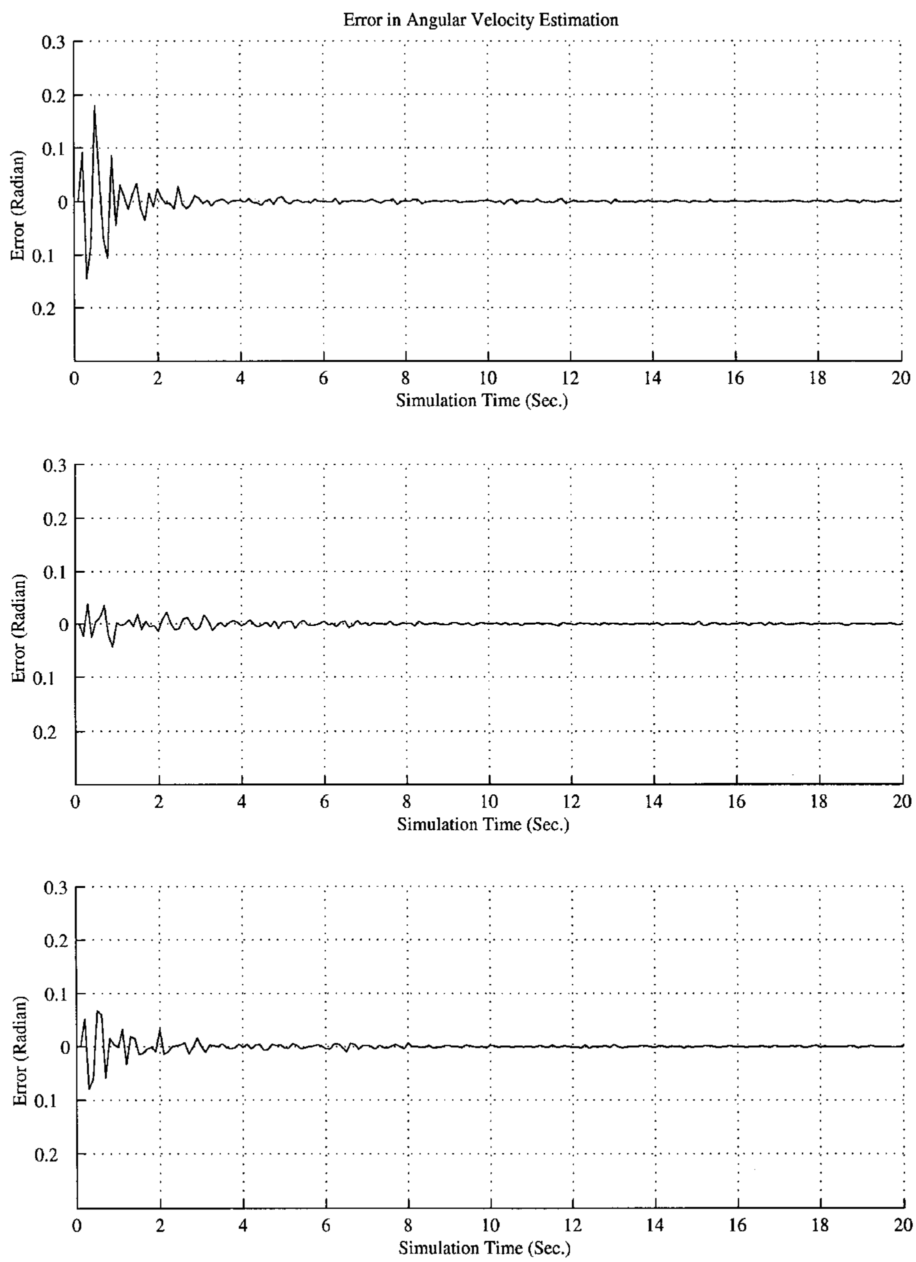


Fig. 6 Angular velocity estimation error when the time step is 0.1 s and measurement noise variance is 0.01.

attitude solution obtained in the preceding step, i.e., to determine $\hat{R}_{i+1}$, $\hat{R}_i$ is used to begin the iteration at time $t_{i+1}$. However, when the three-dimensional identity matrix $I$ is used as the starting point of the iteration, the estimation error as shown in the column labeled "static measurement-only minimization (2)" is larger than those for the preceding initial condition. These results demonstrate the importance of the choice of starting point, because the objective function $\mathrm{tr}(QR^TNR - 2WR)$ will usually have several local minima.

From the results we can infer that the accuracy of the dynamics-based algorithm depends as much on the time step as the measurement noise variance. In particular, the performance in accuracy is as good as the rate sensor-aided case for the case of a 0.1-s time step.

## Conclusions

In this paper the attitude-determination problem is defined for a smoothly rotating satellite with known dynamics. To solve the problem, a new attitude-determination algorithm, which considers the dynamics of the satellite as well as accumulated GPS measurement data, is proposed.

When a rate sensor is used with the GPS receiver, the estimation accuracy of the dynamics-based algorithm is not affected significantly by the length of the time step, and the optimal attitude solution with an estimation error of less than 1 deg can be obtained for time steps up to 5 s or longer. Rather, it is the choice of starting point in the optimization procedure that is important as the length of the time step increases. When the GPS measurement data or the attitude solution are not available for a certain time interval, the suboptimal attitude solution greatly improves the performance of the dynamics-based attitude-determination algorithm.

In the event that angular velocity measurements are not available, we can use the dynamics-based attitude-determination algorithm in conjunction with a cubic-spline interpolation algorithm on SO(3) to estimate the angular velocity provided that the satellite revolves smoothly, i.e., the trajectory of the angular velocity is a smooth curve on $\mathbb{R}^3$. For reasonably slowly varying rotations, when the time step is 0.1 s, the dynamics-based attitude-determination algorithm enables us to estimate the attitude of a satellite with almost the same accuracy as the rate sensor-aided case. The attitude estimation error in roll-pitch-yaw angles in degrees is smaller than that of the static measurement-only minimization by nearly an order of magnitude. If the capability of GPS receivers improve enough so as to generate GPS measurement data with a sufficiently high rate, our results suggest that a satellite with only a GPS receiver can determine the attitude and the angular velocity with very high accuracy using the proposed dynamics-based algorithm.

For our simulation results the attitude-determination algorithm considers only the noise in the measurement model and assumes the control torque of zero. However, for nonzero control torque the dynamics model can be influenced by the process noise added to the control torque; this can require a more rigorous characterization of noise processes in SO(3) and remains a topic for further research.

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